

Efficient Algorithms for Dualizing Large-Scale Hypergraphs

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Abstract. A hypergraph \mathcal{F} is a set family defined on vertex set V . The dual of \mathcal{F} is the set of minimal subsets H of V such that $F \cap H \neq \emptyset$ for any $F \in \mathcal{F}$. The computation of the dual is equivalent to many problems, such as minimal hitting set enumeration of a subset family, minimal set cover enumeration, and the enumeration of hypergraph transversals. Although many algorithms have been proposed for solving the problem, to the best of our knowledge, none of them can work on large-scale input with a large number of output minimal hitting sets. This paper focuses on developing time- and space-efficient algorithms for solving the problem. We propose two new algorithms with new search methods, new pruning methods, and fast techniques for the minimality check. The computational experiments show that our algorithms are quite fast even for large-scale input for which existing algorithms do not terminate in a practical time.

1 Introduction

A *hypergraph* \mathcal{F} is a subset family defined on a vertex set V , that is, each element (called *hyperedge*) F of \mathcal{F} is a subset of V . The hypergraph is a generalization of a graph so that edges can have more than two vertices. A *hitting set* is a subset H of V such that $H \cap F \neq \emptyset$ for any hyperedge $F \in \mathcal{F}$. A hitting set is called *minimal* if it includes no other hitting set. The *dual* of a hypergraph is the set of all minimal hitting sets. The *dualization* of a hypergraph is to construct the dual of a given hypergraph.

Dualization is a fundamental problem in computer science, especially in machine learning, data mining, and optimization, etc. It is equivalent to (1) the minimal hitting set enumeration of given subset family, (2) minimal set cover enumeration of given set family, (3) enumeration of hypergraph transversal, (4) enumeration of minimal subsets that are not included in any of the given set family, etc. One of the research goals is to clarify the existence of a polynomial time algorithm for solving the problem. The size of dual can be exponential in the input hypergraph, thus the polynomial time algorithm for dualization usually means an algorithm running in time polynomial to the input size and the output size. Although Kachian et al.[6] developed a quasi-polynomial time algorithm which runs in $O(N^{\log N})$ time, where N is the input size plus output size, the existence of a polynomial time algorithm is still an open question.

From the importance of dualization in its application areas, a lot of research has aimed at algorithms that terminate in a short time on real world data. The size of the dual can be exponential, but in practice, it is huge but not intractable. Thus, practically efficient algorithms aim to take a short time for each minimal hitting set. Reduction of the search space was studied as a way to cope with this problem [4, 6–8, 12, 15]. Finding a minimal hitting set is easy; one removes vertices one by one unless each has an empty intersection with some hyperedges. However, finding exactly all minimal hitting sets is not easy; we have to check a great many vertex subsets that can be minimal hitting sets. The past studies have succeeded in reducing the search space, but the computational cost was substantial, hence the current algorithms may take a long time when the size of the dual is large.

In this paper, we focus on developing an efficient computation for the case of large-scale input data with a large number of minimal hitting sets. We looked at the disadvantages of the existing methods and devised new algorithms to eliminate them.

- *breadth-first search*: A popular search method for dualization is hill climbing such that the algorithm starts from the emptyset, and recursively adds vertices one by one until it reaches minimal hitting sets. The minimal hitting sets already found are stored in memory and used to check the minimality. This minimality check is popular, but its memory usage is so inefficient so that we cannot solve a problem with many minimal hitting sets. We alleviate this disadvantage by using a depth-first search algorithm with the use of the new minimality check algorithm explained below. The algorithm proposed in [8, 9] uses a depth-first search, but its minimality check takes a long time on large hypergraphs.
- *minimality check*: The time for the minimality check in a breadth-first search is short when the hitting sets to be checked are small on average, but will be long for larger hitting sets (such as size 20 or larger). We alleviate this disadvantage by using a new algorithm that does not need the hitting sets that have already been found. We introduce a new concept, called the *critical hyperedge*, that characterizes the minimality of hitting sets. Computing and updating critical hyperedges can be done in a short time, thus we can efficiently check the minimality in a short time.
- *pruning*: Several algorithms use pruning methods to reduce the search space, but our experiments show that these pruning methods are not sufficient. We propose a simple but efficient pruning method. We introduce a lexicographic depth-first search, and thereby remove vertices that can never be used and prune branches without necessary vertices. The pruning drastically reduces the computation time.
- *sophisticated use of simple data structures*: Not many studies have mentioned the data structures or how to use them efficiently, despite this being a very important consideration to reduce the computation time. We use both the adjacency matrix (characteristic vectors of hyperedges) and doubly linked lists to speed up the operations of taking intersections and set differences.

This accelerates the computation time in extremely sparse, extremely dense (use complement as input), non-small minimal hitting sets (over 10 vertices) cases.

The paper is organized as follows. In the following subsections, we explain the related work and related problems. Section 2 is for preliminaries, and Section 3 describes the existing algorithms. We describe our new algorithms in Section 4 and show the results of computational experiments in Section 5. We conclude the paper in Section 6.

1.1 Related Work

There have been several studies on the dualization problem, of which we shall briefly review the DL, BMR, KS and HBC algorithms. These algorithms are classified into two types according to their structure; improved versions of the Berge algorithm[2], and hill-climbing algorithms. The Berge algorithm updates the set of minimal hitting sets iteratively, by adding hyperedges one by one to the current partial hypergraph. DL, BMR and KS are the algorithms of this type, and HBC is the hill-climbing type. The candidates for minimal hitting sets are generated by gathering vertices one-by-one until a minimality condition is violated. When a candidate becomes a hitting set, it is a minimal hitting set. The HBC algorithm does this operation in a breadth-first manner.

The DL algorithm, proposed by Dong and Li [4], is a border-differential algorithm for data mining. The main difference from the Berge algorithm is that it avoids generating non-minimal hitting sets by increasing the problem size incrementally. The DL algorithm starts from an empty hypergraph and adds a hyperedge iteratively while updating the set of minimal hitting sets. The sizes of the intermediate sets of minimal hitting sets are likely smaller than that of the original hypergraph, thus we can expect that there will be no combinatorial explosion. Experiments on two small UCI datasets [14] have shown that the DL algorithm is much faster than their previous algorithm and the level-wise hill climbing algorithm.

In general, the Berge algorithm and DL algorithm are very useful when the hypergraph has few hyperedges, but for large hypergraphs, it may take a long time because of many updates. The BMR algorithm, proposed by Bailey et al. [5], starts from a hypergraph with few vertices with hyperedges restricted to the vertex set (the vertices not in the current vertex set are removed from the hyperedges). The hyperedges grow as the vertex set increases. The BMR algorithm first uses the Berge algorithm to solve the problem of the initial hypergraph, and then it updates the minimal hitting sets. Note that Hagen tested a version of the DL algorithm instead of the Berge algorithm [11].

Kavvadias and Stavropoulos's algorithm (KS algorithm) [8, 9] embodies two ideas; unifying the nodes contained in the same hyperedges and depth-first search. These ideas help to reduce the number of intermediate hitting sets and memory usage. To perform a depth-first search, they use a minimality check algorithm that does not need other hitting sets; check whether the removal of each

vertex results a hitting set or not. The KS algorithm uses an efficient algorithm for this task.

Hebert et al. proposed a level-wise algorithm (HBC algorithm) [7]. Their algorithm is a hill climbing algorithm which starts from the empty set and adds vertices one by one. It searches the vertex subsets satisfying a necessary condition to be a minimal hitting set, called a “Galois connection”. A vertex subset satisfies the Galois connection if the removal of any of its vertexes decreases the number of hyperedges intersecting with it. The sets satisfying the Galois connection form a set system satisfying the monotone property (independent set system), thus we can perform a breadth-first search in the usual way.

1.2 Related Problems

Dualization has many equivalent problems. We show some of them below.

(1) *minimal set cover enumeration*

For a subset family \mathcal{F} defined on a set E , a set cover S is a subset of \mathcal{F} such that the union of the members of S is equal to E , i.e., $E = \bigcup_{X \in S} X$. A set cover is called minimal if it is included in no other set cover. We consider \mathcal{F} to be a vertex set, and $\mathcal{F}(v)$ to be a hyperedge where $\mathcal{F}(v)$ is the set of $F \in \mathcal{F}$ that include v . Then, for the hyperedge set (set family) $F = \{\mathcal{F}(v) | v \in E\}$, a hitting set of F is a set cover of \mathcal{F} , and vice versa. Thus, enumerating minimal set covers is equivalent to dualization.

(2) *minimal uncovered set enumeration*

For a subset family \mathcal{F} defined on a set E , an uncovered set S is a subset of E such that S is not included in any member of \mathcal{F} . Let $\bar{\mathcal{F}}$ be the complement of \mathcal{F} , which is the set of the complement of members in \mathcal{F} , i.e., $\bar{\mathcal{F}} = \{E \setminus X | X \in \mathcal{F}\}$. S is not included in $X \in \mathcal{F}$ if and only if S and $E \setminus X$ have a non-empty intersection. An uncovered set of \mathcal{F} is a hitting set of $\bar{\mathcal{F}}$, and vice versa, thus the minimal uncovered set enumeration is equivalent to the minimal hitting set enumeration.

(3) *circuit enumeration for independent system*

A subset family \mathcal{F} defined on E is called an independent system if for each member X of \mathcal{F} , any of its subsets is also a member of \mathcal{F} . A subset of E is called independent if it is a member of \mathcal{F} , and dependent otherwise. A circuit is a minimal dependent set, i.e., a dependent set which properly contains no other dependent set. When an independent system is given by the set of maximal independent sets of \mathcal{F} , then the enumeration of circuits of \mathcal{F} is equivalent to the enumeration of uncovered sets of \mathcal{F} .

(4) *Computing negative border from positive border*

A function is called Boolean if it maps subsets in 2^V to $\{0, 1\}$. A Boolean function B is called monotone (resp., anti-monotone) if it for any set X with $B(X) = 0$ (resp., $B(X) = 1$), any subset X' of X satisfies $B(X') = 0$ (resp., $B(X') = 1$).

For a monotone function B , a subset X is called a positive border if $B(X) = 0$ and no its proper superset Y satisfies $B(Y) = 0$, and is called a negative border if $B(X) = 1$ and no its proper subset Y satisfies $B(Y) = 1$. When we are given a Boolean function by the set of positive borders, the problem is to enumerate all of its negative borders. This problem is equivalent to dualization, since the problem is equivalent to uncovered set enumeration.

(5) DNF to CNS transformation

DNF is a formula whose clauses are composed of literals connected by “or” and whose clauses are connected by “and”. CNF is a formula whose clauses are composed of literals connected by “and”, and whose clauses are connected by “or”. Any formula can be represented as a DNF formula and a CNF formula. Let D be a DNF formula composed of variables x_1, \dots, x_n and clauses C_1, \dots, C_m . A DNF/CNF is called monotone if no clause contains a literal with “not”. Then, S is a hitting set of the clauses of D if and only if the assignment obtained by setting the literals in S to true gives a true assignment of D . Let H be a minimal CNF formula equivalent to D . H has to include any minimal hitting set of D as its clause, since any clause of H has to contain at least one literal of any clause of D . Thus, a minimal CNF equivalent to D has to include all minimal hitting sets of D . For the same reason, computing the minimal DNF from a CNF is equivalent to dualization.

2 Preliminaries

A *hypergraph* \mathcal{F} is a subset family $\{F_1, \dots, F_m\}$ defined on a vertex set V , that is, each element (called *hyperedge*) F of \mathcal{F} is a subset of V . The hypergraph is a generalization of a graph so that edges can contain more than two vertices. A subset H of V is called a *vertex subset*. A *hitting set* is a vertex subset H such that $H \cap F \neq \emptyset$ for any hyperedge $F \in \mathcal{F}$. A hitting set is called *minimal* if it includes no other hitting set. The *dual* of a hypergraph is the hypergraph whose hyperedge set is the set of all minimal hitting sets, and it is denoted by $dual(\mathcal{F})$. For example, when $V = \{1, 2, 3, 4\}$, $\mathcal{F} = \{\{1, 2\}, \{1, 3\}, \{2, 3, 4\}\}$, $\{1, 3, 4\}$ is a hitting set but not minimal, and $\{2, 3\}$ is a minimal hitting set. $dual(\mathcal{F})$ is $\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}\}$. It is known that $\mathcal{F} = dual(dual(\mathcal{F}))$ if no $F, F' \in \mathcal{F}$ satisfy $F \subset F'$. The *dualization* of a hypergraph is to construct the dual of the given hypergraph.

$|\mathcal{F}|$ denotes the number of hyperedges in \mathcal{F} , that is m , and $||\mathcal{F}||$ denotes the sum of the sizes of hyperedges in \mathcal{F} , respectively. In particular, $||\mathcal{F}||$ is called the *size* of \mathcal{F} . \mathcal{F}_i denotes the hypergraph composed of hyperedges $\{F_1, \dots, F_i\}$. For $v \in V$, let $\mathcal{F}(v)$ be the set of hyperedges in \mathcal{F} that includes v , i.e., $\mathcal{F}(v) = \{F | F \in \mathcal{F}, v \in F\}$. For vertex subset S and vertex v , we respectively denote $S \cup \{v\}$ and $S \setminus \{v\}$ by $S \cup v$ and $S \setminus v$.

We introduce the new concept *critical hyperedge* in the following. For a vertex subset $S \subseteq V$, $uncov(S)$ denotes the set of hyperedges that do not intersect with S , i.e., $uncov(S) = \{F | F \in \mathcal{F}, F \cap S = \emptyset\}$. S is a hitting set if and only if

$uncov(S) = \emptyset$. For a vertex $v \in S$, a hyperedge $F \in \mathcal{F}$ is said to be *critical* for v if $S \cap F = \{v\}$. We denote the set of all critical hyperedges for v by $crit(v, S)$, i.e., $crit(v, S) = \{F \in \mathcal{F} \mid S \cap F = \{v\}\}$. Suppose that S is a hitting set. If v has no critical hyperedge, every $F \in \mathcal{F}$ includes a vertex in S other than v , thus $S \setminus v$ is also a hitting set. Therefore, we have the following property.

Property 1. S is a minimal hitting set if and only if $uncov(S) = \emptyset$, and $crit(v, S) \neq \emptyset$ holds for any $v \in S$.

If $crit(v, S) \neq \emptyset$ for any $v \in S$, we say that S satisfies the *minimality condition*. Our algorithm updates $crit$ to check the minimality condition quickly, by utilizing the following lemmas. Let us consider an example of $crit$. Suppose that $\mathcal{F} = \{\{1, 2\}, \{1, 3\}, \{2, 3, 4\}\}$, and the hitting set S is $\{1, 3, 4\}$. We can see that $crit(1, S) = \{\{1, 2\}\}$, $crit(3, S) = \emptyset$, $crit(4, S) = \emptyset$, thus S is not minimal, and we can remove either 3 or 4. For $S' = \{1, 3\}$, $crit(1, S') = \{\{1, 2\}\}$, $crit(3, S') = \{\{2, 3, 4\}\}$, thus S' is a minimal hitting set. The following lemmas are the keys to our algorithms.

Lemma 1. *For any vertex subset S , $v \in S$ and $v' \notin S$, $crit(v, S \cup v') = crit(v, S) \setminus \mathcal{F}(v')$. Particularly, $crit(v, S \cup v') \subseteq crit(v, S)$ holds.*

Proof. For any $F \in crit(v, S)$, $(S \cup v') \cap F = \{v\}$ holds if F is not in $\mathcal{F}(v')$, and thus it is included in $crit(v, S \cup v') \setminus \mathcal{F}(v)$. Conversely, $(S \cup v') \cap F = \{v\}$ holds for any $F \in crit(v, S \cup v')$. This means that $S \cap F = \{v\}$, and $F \in crit(v, S)$. \square

Lemma 2. *For any vertex subset S and $v' \notin S$, $crit(v', S \cup v') = uncov(S) \cap \mathcal{F}(v')$.*

Proof. Since any hyperedge not in $uncov(S)$ has a non-empty intersection with S , F can never be a critical hyperedge for v' . Any critical hyperedge for v' includes v' thus, we can see that $crit(v', S) \subseteq uncov(S) \cap \mathcal{F}(v')$. Conversely, for any hyperedge F included in $uncov(S) \cap \mathcal{F}(v')$, $F \cap (S \cup v') = \{v'\}$, thereby $uncov(S) \cap \mathcal{F}(v') \subseteq crit(v', S)$. Hence, the lemma holds. \square

The next two lemmas follow directly from the above.

Lemma 3. *[7] If a vertex subset S satisfies the minimality condition, any of its subsets also satisfy the minimality condition, i.e., the minimality condition satisfies the monotone property.*

Lemma 4. *[7] If a vertex subset S does not satisfy the minimality condition, S is not included in any minimal hitting set. In particular, any minimal hitting set S is maximal in the set system composed of vertex subsets satisfying the minimality condition.*

Lemma 5. *For any vertex subset S , $\sum_{v \in S} |crit(v, S)| \leq |\mathcal{F}|$.*

Proof. From the definition of the critical hyperedge, any hyperedge $F \in \mathcal{F}$ can be critical for at most one vertex. Thus, the lemma holds. \square

3 Existing Algorithms

This section is devoted to explaining the framework of the existing algorithms related to our algorithms: DL algorithm, KS algorithm, and HBC algorithm. The DL algorithm starts by computing $dual(\mathcal{F}_1)$ and then iteratively computes $dual(\mathcal{F}_i)$ from $dual(\mathcal{F}_{i-1})$. For any $S \in dual(\mathcal{F}_i)$, either $S \in dual(\mathcal{F}_{i-1})$ holds, or $S \setminus v \in dual(\mathcal{F}_{i-1})$ holds for $\{v\} = S \cap F_i$. Note that when $S \in dual(\mathcal{F}_i)$ is not in $dual(\mathcal{F}_{i-1})$, $S \cap F_i$ is composed of exactly one vertex, since $crit(v, S)$ must be $\{F_i\}$. However, for any $S \in dual(\mathcal{F}_{i-1})$, $S \in dual(\mathcal{F}_i)$ if $S \cap F_i \neq \emptyset$. When $S \cap F_i = \emptyset$, $S \cup v$ with $v \in F_i$ may be in $dual(\mathcal{F}_i)$. The algorithm is as follows.

ALGORITHM DL ($\mathcal{F} = \{F_1, \dots, F_m\}$)

1. $\mathcal{D}_0 := \{\emptyset\}$
2. **for** $i := 1$ **to** m
3. $\mathcal{D}_i := \emptyset$
4. **for each** $S \in \mathcal{D}_{i-1}$ **do**
5. **if** $S \cap F_i \neq \emptyset$ **then** insert S to \mathcal{D}_i
6. **else for each** $v \in F_i$ **do**
7. **if** no $S' \in \mathcal{D}_{i-1}$ satisfies $S' \neq S$ and $S' \subseteq S \cup v$ **then** insert $S \cup v$ to \mathcal{D}_i
8. **end for**
9. **end for**
10. **end for**

After the computation, \mathcal{D}_m is $dual(\mathcal{F})$. Line 7 is for checking whether $S \cup v$ is in \mathcal{D}_i or not by looking for a hitting set included in $S \cup v$. This needs basically $O(\sum \|\mathcal{D}_{i-1}\|)$ time and is a bottleneck computation of the algorithm. This part requires all of \mathcal{D}_i memory, thus we need to perform a breadth-first search. Kavadias and Stavropoulos[8, 9] proposed a depth-first version of this algorithm. According to the hitting sets generation rule, each hitting set in \mathcal{F}_i is uniquely generated from a hitting set of \mathcal{F}_{i-1} . Thus, starting from each hitting set in \mathcal{F}_1 , we perform this generation rule in a depth-first manner, and visit all the minimal hitting sets of all \mathcal{F}_i . The algorithm does not store each \mathcal{D}_i in memory, and it checks for the minimality of $S \cup v$ by checking whether $S \cup v \setminus f$ is a hitting set or not for each $f \in S$. The algorithm is as follows.

ALGORITHM KS (S, i)

1. **if** $i = m$ **then output** S ; **return**
2. **if** $S \cap F_i \neq \emptyset$ **then call** KS($S, i + 1$)
3. **else for each** $v \in F_i$ **do**
4. **for each** $u \in S$ **do**
5. **if** $S \cup v \setminus u$ is a hitting set **then go to** 8.
6. **end for**
7. **call** KS($S \cup v, i + 1$)
8. **end for**

The bottleneck is also the minimality check on line 5 that basically needs to access all hyperedges in \mathcal{F}_{i-1} .

The number hitting sets that are added a vertex is $|\bigcup_{i=1}^m \mathcal{D}_i \setminus dual(\mathcal{F})| = O(|\bigcup_{i=1}^m \mathcal{D}_i|)$. The algorithms perform the minimality check for each addition, thus roughly speaking, the number of minimality checks in both algorithms is $O(|\bigcup_{i=1}^m \mathcal{D}_i| \times f)$ where f is the average size of hyperedges. For $S \in \mathcal{D}_i$ and $v \in S$, $crit(v, S)$ is non-empty, since v always has a critical hyperedge in \mathcal{F}_i . It implies that any subset S explored by the algorithm satisfies the minimality condition.

The minimality check is usually one of the time-consuming parts of dualization algorithms. The check whether the current vertex subset is a hitting set or not is also a time consuming part, but it can be done by updating *uncov*, thus for almost all vertex subsets to be operated on, its cost is much smaller than the minimality check. Therefore, the number of minimality checks would be a good measure of the efficiency of the search strategy. Here, we define the *search space* of an algorithm by the set of vertex subsets that are checked the minimality. The size of the search space is equal to the number of executed minimality checks.

The cost for the minimality check increases with $|\mathcal{D}_i|$, for the DL algorithm, and with S and $||\mathcal{F}||$ for the KS algorithm. Thus, the DL algorithm will be faster when the $dual(\mathcal{F})$ is small, whereas the KS algorithm will be faster when $||\mathcal{F}||$ is small and S is small on average.

The HBC algorithm is a kind of branch and bound algorithm. It starts from the emptyset, and chooses elements one by one. For each element v , it generates two recursive calls concerned with a choice; add v to the current vertex subset, and do not add it. When the current vertex subset becomes a hitting set, it checks the minimality, and outputs it if minimal. To speed up the computation, the algorithm prunes branches through the use of the so called Galois condition. The Galois condition for S and $v \notin S$ is $|uncov(S)| = |uncov(S \cup v)|$, and when it holds, $S \cup v$ is never included in a minimal hitting set, thus we can terminate the recursive call with respect to $S \cup v$. The Galois condition is equivalent to our minimality condition, since it is equivalent to $crit(v, S \cup v) = \emptyset^1$. The algorithm is written as follows.

ALGORITHM HBC ($\mathcal{F} = \{F_1, \dots, F_m\}$)

1. $\mathcal{D}_0 := \{\emptyset\}$; $i = 0$
2. **while** $\mathcal{D}_i \neq \emptyset$
3. **for each** $S \in \mathcal{D}_i$ **do**
4. **if** $uncov(S) = \emptyset$ **then** output S
5. **for each** v larger than maximum vertex in S **do**
6. **if** $S \cup v$ satisfies the Galois condition **then** insert S to \mathcal{D}_i
7. **end for**
8. **end for**
9. **end while**

¹ the Galois condition is proposed in 2007[7], while *crit* is proposed in 2003[12, 15]. The term “minimality condition” first appears in [7].

If the pruning method is only the Galois condition, the vertex subsets to be explored by the algorithm is all the non-hitting sets satisfying the minimality condition. Thus, the size of search space of HBC algorithm is no less than that of DL algorithm. On contrary, DL and KS algorithms has to update the minimal hitting sets even if they do not change, thus the HBC algorithm has an advantage in this point.

4 New Search Algorithms and Minimality Check

We propose two depth-first search (branch and bound) algorithms for dualization problem. The main differences from the existing algorithms are to use *crit* for the minimality condition check, and pruning methods to avoid searching hopeless branches. The algorithms keep lists *crit*[*u*] and *uncov* representing *crit*(*u*, *S*) and *uncov*(*S*). When the algorithm adds a vertex *v* to *S* and generates a recursive call, it updates *crit*[] and *uncov* by the following algorithm.

Update_crit_uncov (*v*, *crit*[], *uncov*)

1. **for each** $F \in \mathcal{F}(v)$ **do**
2. **if** $F \in \text{crit}[u]$ for a vertex $u \in S$ **then** remove F from *crit*[*u*]
3. **if** $F \in \text{uncov}$ **then** $\text{uncov} := \text{uncov} \setminus F$; $\text{crit}[e] := \text{crit}[e] \cup \{F\}$
4. **end for**

After execution, *crit*[*u*] becomes *crit*(*u*, $S \cup v$). Since each hyperedge F can be critical hyperedge for at most one vertex, we put F on the vertex as a mark and perform step 2 in a constant time. Thus, the time complexity of this algorithm is $O(|\mathcal{F}(v)|)$. Even though this algorithm is simple, we can reduce the time complexity of an iteration of the KS algorithm from $O(|V| \times \|\mathcal{F}\|)$ to $O(\|\mathcal{F}\|)$.

4.1 Reverse Search Algorithm

One of our algorithms is based on the reverse search [1], and it can be regarded as an improved version of the KS algorithm. Let $\mathcal{S} = \bigcup_{i=1}^m \mathcal{F}_i$, that is the set of vertex subsets that are operated by KS algorithm. Let us denote the minimum i such that $F_i \in \text{crit}(v, S)$ by $\text{min_crit}(v, S)$, and the minimum i such that $F_i \in \text{uncov}(S)$ by $\text{min_uncov}(S)$. $\text{min_crit}(v, S)$ (resp., $\text{min_uncov}(S)$) is defined as $m + 1$ if *crit*(*v*, *S*) (resp., *uncov*(*S*)) is empty. Using these terms, we give a characterization of \mathcal{S} .

Lemma 6. $S \neq \emptyset$ belongs to \mathcal{S} if and only if $\text{min_crit}(v, S) < \text{min_uncov}(S)$ holds for any $v \in S$.

Proof. Suppose that $S \in \mathcal{S}$, thus $S \in \mathcal{F}_i$ for some i . We can see that $\text{min_uncov}(S) > i$, *crit*(*v*, *S*) includes a hyperedge $F_j \in \mathcal{F}$ with $i < j$, and thus $\text{min_crit}(v, S) < i$ for any v . Thus, $\text{min_crit}(v, S) < \text{min_uncov}(S)$ holds for any $v \in S$.

Conversely, suppose that $\text{min_crit}(v, S) < \text{min_uncov}(S)$ holds for any $v \in S$. Then, we can see that $\text{crit}(v, S) \neq \emptyset$ for any $v \in S$ because $\text{min_crit}(v, S) <$

$m + 1$. Let $i = \min_uncov(S) - 1$. Note that $i \leq m$. We can then see that S is a hitting set of \mathcal{F}_i and $\min_crit(v, S) \leq i$. This in turn implies that S is a minimal hitting set in \mathcal{F}_i , and thus, it belongs to \mathcal{S} . \square

For $S \in \mathcal{S}$, $\min_crit(S)$ is the minimum index i such that S is a minimal hitting set of \mathcal{F}_i , i.e., $\min_crit(S) = \max_{v \in S} \{\min_crit(v, S)\}$. We define the parent $P(S)$ of S by $S \setminus v$, where v is the vertex such that $\min_crit(v, S) = \min_crit(S)$. Since any F_i is critical for at most one vertex, $\min_crit(S)$ and the parent are uniquely defined. The parent-child relation given by this definition is acyclic, thus forms a tree spanning all the vertex subsets in \mathcal{S} and rooted at the emptyset. Our algorithm performs a depth-first search on this tree starting from the emptyset. This kind of search strategy is called reverse search[1].

This search strategy is essentially equivalent to KS algorithm if we skip all redundant iteration in which we add no vertex to the current vertex subset. In a straightforward implementation of KS algorithm, we have to iteratively compute the intersection of F_i and the current vertex subset S until we meet the F_i that does not intersect with S . When $uncov(S)$ is not so large, it takes long time. Particularly, when $uncov(S) = \emptyset$, we may spend $\Theta(|\mathcal{F}|)$ time. On contrary, in our strategy, we have only to maintain $uncov(S)$, that is much lighter.

The depth-first search starts from the emptyset. When it visits a vertex subset S , it finds all children of S iteratively and generates a recursive call for each child. In this way, we can perform a depth-first search only by finding children of the current vertex subset. The way to find the children is shown in the following lemma.

Lemma 7. *Let $S \in \mathcal{S}$ and $i = \min_uncov(S)$. A vertex subset S' is a child of S if and only if*

- (1) $i < m + 1$
- (2) $S' = S \cup v$ for some $v \in F_i$, and
- (3) $\min_crit(v', S') < i$ holds for any $v' \in S$.

Proof. Suppose that S' is a child of S . We can see that $uncov(S)$ is not empty, and thus (1) holds. From the definition of the parent, S is obtained from S' by removing a vertex v from S' . From $\min_crit(S') < \min_uncov(S')$ and $uncov(S) = uncov(S') \cup crit(v, S')$, we obtain $\min_crit(v, S') = \min_crit(S') = \min_uncov(S)$. This means that $F_i \in crit(v, S')$, and thus (2) holds. This equation also implies that (3) holds.

Suppose that S' is a vertex subset satisfying (1), (2) and (3). From (2), we see that $\min_crit(v, S') = i$. Since $uncov(S') > i$, this together with (3) implies that S' satisfies the conditions in Lemma 6 and thereby is included in \mathcal{S} . $\min_crit(v, S') = i$ and (3) leads to $\min_crit(S') = i$ and $P(S') = S' \setminus v = S$. Note that condition (1) guarantees the existence of F_i given condition (2), thus it is implicitly used in the proof. \square

From Lemma 7, we can find all children of S by adding each vertex $v \in F_i$ to S , and checking (3). This can be done in a short time by updating $crit$. The algorithm is as follows.

global variable: $crit[]$, $uncov$

ALGORITHM **RS** (S)

1. **if** $uncov = \emptyset$ **then output** S ; **return**
2. $i := \min\{j | F_j \in uncov\}$
3. **for each** $v \in F_i$ **do**
4. call `Update_crit_uncov` ($v, crit[], uncov$)
5. **if** $\min\{t | F_t \in crit[f]\} < i$ **for each** $f \in S$ **then call** **RS**(S')
6. recover the change to $crit[]$ and $uncov$ done in 4
7. **end for**

Theorem 1. *Algorithm RS enumerates all minimal hitting sets in $O(|\mathcal{F}| \times |S|)$ time and $O(|\mathcal{F}|)$ space.*

Proof. Since the parent-child relationship induces a rooted tree spanning all vertex subsets in \mathcal{S} , the algorithm certainly enumerates all vertex subsets in \mathcal{S} . Since any minimal hitting set is included in \mathcal{S} , all minimal hitting sets are found by the algorithm. The update of $crit[]$ and $uncov$ is done in $O(|F(v)|)$ time, thus an iteration of the algorithm takes $O(|\mathcal{F}|)$ time. In total, the algorithm takes $O(|\mathcal{F}| \times |\mathcal{S}|)$ time.

The algorithm requires extra memory for storing $crit[]$ and $uncov$ and for memorizing the hyperedges removed in step 4. Since $crit[]$ and $uncov$ are pairwise disjoint, the total memory for $crit$ and $uncov$ is $O(m)$. If a hyperedge is removed from a list, it will not be removed again in the deeper levels of the recursion, from the monotonicity of $crit$. Thus, it also needs $O(m)$ memory. The most memory is for $\mathcal{F}(v)$ of each v , and takes $O(|\mathcal{F}|)$ space. \square

pruning method Suppose that in an iteration we are operating on a vertex subset S , and have confirmed that $S \cup v$ does not satisfy the minimality condition. From Lemma 3, we observe that $S' \cup v$ does not satisfy the minimality condition if $S \subseteq S'$. This means that in the recursive call generated by the iteration with respect to S , we do not have to care about the addition of v , thus we remove v from the candidate list for addition during the recursive call. This condition also holds when $S \cup v$ is a minimal hitting set, since no superset of a minimal hitting set satisfies the minimality condition. We call the vertex v satisfying one of these conditions *violating*.

We can apply this pruning method to the RS algorithm by finding all violating vertices before step 3 and can output all minimal hitting sets $S \cup v$ found in the process. We then execute the loop from step 3 to step 7 only for non-violating vertices, so that we can avoid unnecessary recursive calls.

4.2 Depth-first Search Algorithm

This subsection described a simple hill-climbing depth-first search algorithm, whose search space is contained in that of the HBC algorithm. We start from $S = \emptyset$, and add vertices to S recursively unless the minimality condition is violated. To avoid the duplication, we use a list of vertices *CAND* that represents the vertices that can be added in the iteration. The vertices not included in *CAND*

will not be added, even if the addition satisfies the minimality condition, i.e., the iteration given S and $CAND$ enumerates all minimal hitting sets including S and included in $S \cup CAND$ by recursively generating calls.

Suppose that an iteration is given S and $CAND$, and without loss of generality $CAND = \{v_1, \dots, v_k\}$. For the first vertex v_1 , we make a recursive call with respect to $S \cup v_1$, with $CAND = CAND \setminus v_1$, to enumerate all minimal hitting sets including $S \cup v_1$. After the termination of the recursive call, we generate a recursive call for $S \cup v_2$. To avoid finding the minimal hitting sets including v_1 , we give $CAND \setminus \{v_1, v_2\}$ to the recursive call. In this way, for each vertex v_i , we generate a recursive call with $S \cup v_i$ and $CAND = \{v_{i+1}, \dots, v_k\}$. This search strategy is common to many algorithms for enumerating members in a monotone set system, for example clique enumeration [13]. That is, its correctness has already been proved.

Next, let us describe a pruning method coming from the necessary condition to be a hitting set. Suppose that an iteration is given S and $CAND$, and let F be a hyperedge in $uncov(S)$. We can see that any minimal hitting set including S has to include at least one vertex in S . Thus, we have to generate recursive calls with respect to vertices in $CAND \cap F$, but do not have to do so for vertices in $CAND \setminus F$.

In the RS algorithm, we have to find all violating vertices before generating recursive calls. In contrast, we can omit this step from our DFS algorithm. Suppose that an iteration is given S and $CAND$, and is going to generate recursive calls with respect to vertices in $v_1, \dots, v_k \in F \cap CAND$. Then, we first set $CAND$ to $CAND \setminus \{v_1, \dots, v_k\}$. If v_k is not a violating vertex, we generate a recursive call for $S \cup v_k$, and add v_k to $CAND$. If v_k is a violating vertex, we do not add v_k to $CAND$. In this way, when we generate a recursive call with respect to $S \cup v_h$, all violating vertices $v_j, j > h$ have already been found, thus there is no need to find all them at the beginning. The algorithm is described as follows.

global variable: $crit[]$, $uncov$, $CAND$

ALGORITHM **DFS** (S)

1. **if** $uncov = \emptyset$ **then output** S ; **return**
2. choose a hyperedge F from $uncov$;
3. $C := CAND \cap F$; $CAND := CAND \setminus C$
4. **for each** $v \in C$ **do**
5. call $Update_crit_uncov(v, crit[], uncov)$
6. **if** $crit(f, S') \neq \emptyset$ for each $f \in S$ **then call** **DFS**($S \cup v$); $CAND := CAND \cup v$
7. recover the change to $crit[]$ and $uncov$ done in 5
8. **end for**

Similar to the case of the RS algorithm, the computation time of an iteration is bounded by $O(|\mathcal{F}|)$.

4.3 Implementation Issues

This section is devoted to the computational techniques for improving efficiency. Our data structure for representing hyperedges and $\mathcal{F}(v)$ is an array list in which the IDs of vertices or hyperedges are stored. Using array list fastens the set operations with respect to $\mathcal{F}(v)$ and list vertices in a hyperedge. The data structure for *crit* and *uncov* is a doubly linked list. In each iteration, we remove some hyperedge IDs from these lists and reinsert them after the termination of a recursive call. A doubly linked list is a good data structure for these operations, as it preserves the order of IDs in the list.

4.4 Using the Adjacency Matrix for Set Operations

When two subsets S and S' are represented by lists of their including elements, the set operations such as intersection and set difference need $O(|S| + |S'|)$ time. However, when we have the characteristic vectors of S and S' , we can do better. The characteristic vector of S is a vector whose i th element is one if and only if i is included in S . To take the intersection, we scan S (or S') with the smaller size, and choose the elements included in S' (or S). This check can be done in $O(1)$ time with using the characteristic vector of S' , thus the computation time is reduced to $O(\min\{|S|, |S'|\})$. For computing $S \setminus S'$, we remove their intersection from S , thus the computation time is also the same.

Our algorithms take intersection of (*crit* and *uncov*) and $F(v)$. Updating the characteristic vectors of (*crit* and *uncov*) uses $O(|\mathcal{F}|)$ memory and does not increase the time complexity. The characteristic vectors of $F(v)$ for each v requires a lot of memory to store, thus we use it only when $||\mathcal{F}||$ is larger than $n \times |\mathcal{F}|/64$, i.e., \mathcal{F} is dense. Note that in our experiments, all instances satisfied this condition.

4.5 Choosing the Smallest Hyperedge

In the DFS algorithm, we can choose arbitrary hyperedge in *uncov* as F , for restricting the vertices to be added. We choose a hyperedge including the smallest number of vertices which have not been pruned, so that the number of recursive calls generated will be small. Counting such vertices in each hyperedge in *uncov* may take time longer than the case just choosing one arbitrary, but our preliminary experiments showed that it reduced the computation time almost in half.

4.6 Pruning Only a Restricted Set of Items

The pruning method described above can be applied to any vertex. However, applying it to all possible vertices may take a long time compared with other parts of an iteration. Sometimes it occurs that pruning takes a long time but only few branches are pruned. Thus, to make the computation time stable, we prune only the vertices in $CAND \cap F$, which are the vertices to be added to the current solution. This takes a time proportional to the time spent by an iteration, thus it never needs a long time.

4.7 Inputting the Complement of the Hypergraph

In some instances, \mathcal{F} is quite dense, e.g., over 95% of vertices are included in many $F \in \mathcal{F}$. This occurs when the data has no clear structure and has many minimal hitting sets. We can often find such instances in practice, such as in minimal infrequent vertex subset mining from maximal frequent vertex subsets. In such cases, the instance itself takes up a lot of memory, and needs a long time to be operated on. Here, we can reduce the computation time by using the complement.

The complement version of our algorithm inputs the complement of each $F \in \mathcal{F}$. The operations of each iteration change so that the vertices to be added are vertices not in F , and taking difference in the *crit* update changes to taking the intersection. This substantially reduces the computation time, since we have to access only a small number of vertices/hyperedges. In our experiments, we found that this idea works well for very dense datasets.

5 Computational Experiments

In this section, we show the results of our computational experiments comparing our algorithms with the existing algorithms.

5.1 Codes and Environments

Our algorithms are implemented in C, without any sophisticated library such as binary tree. Existing algorithms are implemented in C++ by using the vector class in STL. KS algorithm and Fredman Khachiyan algorithm (BEGK[3, 16]) are given by the authors. All tests were performed on a 3.2 GHz Core i7-960 with a Linux operating system with 24GB of RAM memory. Note that none of the implementations used multi-cores. The codes and the instances are available at the author's Web cite (<http://research.nii.ac.jp/uno/dualization.html>).

5.2 Problem Instances

We prepared several instances of problems in several categories as follows. The first category consists of randomly generated instances. Each hyperedge includes a vertex i with probability p . The sizes and the probabilities are listed below.

The instances in the second category were generated by the dataset “connect-4” taken from the UCI Machine Learning Repository [14]. Connect-4 is a board game, and each row of the dataset corresponds to a minimal winning/losing stage of the first player, and a minimal hitting set of a set of winning stages is a minimal way to disturb winning/losing plays of the first player. From the dataset of winning/losing stages, we took the first m rows to make problem instances of different sizes.

The third instances are generated from the frequent itemset (pattern) mining problem. An itemset is a hyperedge in our terminology. For a set family \mathcal{F} and

a support threshold σ , an itemset is called *frequent* if it is included in at least σ hyperedges, and *infrequent* otherwise. A frequent itemset included in no other frequent itemset is called a *maximal frequent itemset*, and an infrequent itemset including no other infrequent itemset is called a *minimal infrequent itemset*. A minimal infrequent itemset is a minimal itemset included in no maximal frequent itemset, and any subset of it is included in at least one maximal frequent itemset. Thus, the dual of the set of the complements of maximal frequent itemsets is the set of minimal infrequent itemsets. The problem instances are generated by enumerating all maximal frequent sets from the datasets “BMS-WebView-2” and “accidents”, taken from the FIMI repository [10]. The profiles of the datasets are listed below.

The fourth instances are used in previous studies [9, 3].

- Matching graph ($M(n)$): a hypergraph with n vertices (n is even) and $n/2$ hyperedges forming a perfect matching, that is, hyperedge F_i is $\{2i - 1, 2i\}$. This instance has few hyperedges but a large number of minimal hitting sets $2^{n/2}$.
- Dual Matching graph ($DM(n)$): it is dual($M(n)$). It has $2^{n/2}$ hyperedges on n nodes. This instance has a large number of hyperedges but a small number of minimal hitting sets $n/2$.
- Threshold graph ($TH(n)$): a hypergraph with n vertices (n is even) and hyperedge set $\{\{i, j\} : 1 \leq i < j \leq n, j \text{ is even}\}$. This instance has a small number of hyperedges $n^2/4$ and a small number of minimal hitting sets $n/2 + 1$.
- Self-Dual Threshold graph ($SDTH(n)$): The hyperedge set of $SDTH(n)$ is given as $\{\{n - 1, n\}\} \cup \{\{n - 1\} \cup E \mid E \in TH(n - 2)\} \cup \{\{n\} \cup E \mid E \in dual(TH(n - 2))\}$. $SDTH(n)$ has the same number of minimal hitting sets as its hyperedges, $(n - 2)^2/4 + n/2 + 1$.
- Self-Dual Fano-Plane graph ($SDFP(n)$): A hypergraph with n vertices and $(k - 2)^2/4 + k/2 + 1$ hyperedges, where $k = (n - 2)/7$. The construction starts with the set of lines in a Fano plane $H_0 = \{\{1, 2, 3\}, \{1, 5, 6\}, \{1, 7, 4\}, \{2, 4, 5\}, \{2, 6, 7\}, \{3, 4, 6\}, \{3, 5, 7\}\}$. Then we set $H = H_1 \cup H_2 \cup \dots \cup H_k$, where H_1, H_2, \dots, H_k are k disjoint copies of H_0 . The dual of H is the hypergraph of all 7_k unions obtained by taking one hyperedge from each of k copies of $H_0(H_1, H_2, \dots, H_k)$. We finally obtain $SDFP(n)$, which is a hypergraph of $1 + 7k + 7k$ hyperedges.

5.3 Differences

Before showing the results, we discuss the difference between the algorithms from the viewpoint of algorithmic structures. Basically, the search space of DL, KS, and our RS algorithms are the same. However, DL and KS check the same hitting sets many times, while RS operates by one hitting set at most once. In addition, our RS has a pruning method, thus the number of hitting sets generated may be decreased. The search spaces of the HBC and DFS algorithms are basically the same, but DFS reduces it by using pruning methods.

For the minimality check, DL, BMR, and HBC algorithms access basically all members in \mathcal{D}_i . Basically, this takes $O(\min\{2^{|S|} \log |\mathcal{F}|, |\mathcal{D}|\})$ time. Some heuristics can reduce the time, but the reduction ratio would be limited. In

contrast, KS takes $O(|S| \times ||\mathcal{F}||)$ time, and RS takes $O(|\mathcal{F}(v)|)$ time. Thus, we can expect that

- DL, BMR, and HBC are faster when there are only a few minimal hitting sets ,
- HBC is faster if the search space of DL is larger than the set of vertex subsets satisfying the minimality condition, for example, in the case that the sizes of minimal hitting sets are quite small
- KS, RS, and DFS are faster when $|\mathcal{F}|$ is small,
- RS is faster than KS when the sizes of minimal hitting sets are not small, and vertex unification (done by KS) does not work.

5.4 Results

Table 1 - 10 compare the computation times. In these tables, $|\mathcal{F}|$ represents the number of hyperedges, $|F|^*$ represents the average size of hyperedges, $|dual(\mathcal{F})|$ represents the number of minimal hitting sets and $|S|^*$ represents the average size of minimal hitting sets. The computation time is in seconds. Furthermore, “-” means that the computation time was more than 1000 seconds, and “fail” implies that the computation did not terminate normally because of a shortage of memory or some error.

Table 1. Computation time on the dataset of winning stage in Connect-4

w	100	200	400	800	1600	3200	6400	12800
BEGK	1.2	5.2	46	55	430	-	-	-
DL	0.005	0.061	1.6	6.2	180	-	-	-
BMR	0.006	0.044	0.52	0.67	17	710	-	-
HBC	33	-	-	-	-	-	-	-
KS	0.021	0.14	1.1	3.2	73	860	-	-
RS	0.001	0.005	0.032	0.078	0.41	4.7	20	83
DFS	0.001	0.006	0.021	0.056	0.27	2.6	11	48
$ \mathcal{F} $	100	200	400	800	1600	3200	6400	12800
$ F ^*$	8	8	8	8	8	8	8	8
$ dual(\mathcal{F}) $	287	1145	6069	11675	71840	459502	1277933	11614885
$ S ^*$	10.70	11.95	14.15	14.84	16.46	17.69	18.67	20.54

The computation time of algorithms which store minimum hitting sets, such as DL and BMR, depends on $|dual(\mathcal{F})|$ and $|S|^*$. On the other hand, the computation time of the depth-first algorithms, such as KS, RS and DFS, depends

Table 2. Computation time on the dataset of losing stage in Connect-4

l	100	200	400	800	1600	3200	6400	12800
BEGK	4.7	51	110	340	-	-	-	-
DL	0.11	6.4	44	210	-	-	-	-
BMR	0.047	2.2	5.1	16	130	-	-	-
HBC	110	-	-	-	-	-	-	-
KS	0.057	2.6	4.6	20	97	-	-	-
RS	0.009	0.052	0.14	0.41	1.6	15	98	420
DFS	0.006	0.044	0.09	0.28	0.94	12	40	180
$ \mathcal{F} $	100	200	400	800	1600	3200	6400	12800
$ F ^*$	8	8	8	8	8	8	8	8
$ dual(\mathcal{F}) $	2341	22760	33087	79632	212761	2396735	4707877	16405082
$ S ^*$	11.19	12.43	13.59	14.62	15.73	17.06	17.41	19.09

Table 3. Computation time on matching graphs **Table 4.** Computation time on dual matching graphs

M	20	24	28	32	36	40
BEGK	0.045	0.72	1.1	4.4	36	fail
DL	0.003	0.012	0.04	0.21	0.89	3.9
BMR	0.003	0.016	0.045	0.19	1.2	5.3
HBC	0.17	2.4	37	520	-	-
KS	0	0.003	0.01	0.044	0.2	0.87
RS	0	0.004	0.013	0.059	0.25	1.1
DFS	0.002	0.006	0.023	0.06	0.26	1.1
$ \mathcal{F} $	10	12	14	16	18	20
$ F ^*$	2	2	2	2	2	2
$ dual(\mathcal{F}) $	2^{10}	2^{12}	2^{14}	2^{16}	2^{18}	2^{20}
$ S ^*$	10	12	14	16	18	20

DM	20	24	28	32	36	40
BEGK	1.4	3.1	8.9	67	fail	fail
DL	0.01	0.054	0.25	1.2	7.1	70
BMR	0.038	0.4	4.2	49	540	-
HBC	0.21	3.3	57	900	-	-
KS	0.012	0.071	0.56	5.6	60	780
RS	0.007	0.054	0.5	4.8	50	-
DFS	0.014	0.075	0.64	6.8	73	-
$ \mathcal{F} $	2^{10}	2^{12}	2^{14}	2^{16}	2^{18}	2^{20}
$ F ^*$	10	12	14	16	18	20
$ dual(\mathcal{F}) $	10	12	14	16	18	20
$ S ^*$	2	2	2	2	2	2

Table 5. Computation time on threshold graphs **Table 6.** Computation time on self-dual threshold graphs

TH	40	80	120	160	200
BEGK	0.28	0.84	2.7	7.5	19
DL	0.004	0.027	0.091	0.24	0.52
BMR	0.009	0.15	0.6	2.6	6.6
HBC	-	-	-	-	-
KS	0.021	0.34	2.5	11	35
RS	0.001	0.003	0.016	0.019	0.048
DFS	0	0.003	0.01	0.026	0.037
$ \mathcal{F} $	400	1600	3600	6400	10000
$ F ^*$	2	2	2	2	2
$ dual(\mathcal{F}) $	21	41	61	81	101
$ S ^*$	29.05	59.02	89.02	119.01	149.01

$SDTH$	42	82	122	162	202
BEGK	0.53	3.3	27	110	310
DL	0.008	0.052	0.2	0.56	1.3
BMR	0.012	0.19	0.87	2.7	7.2
HBC	-	-	-	-	-
KS	0.057	1	6.3	25	74
RS	0.002	0.01	0.017	0.049	0.065
DFS	0.001	0.01	0.025	0.041	0.068
$ \mathcal{F} $	422	1642	3662	6482	10102
$ F ^*$	4.34	4.42	4.45	4.46	4.47
$ dual(\mathcal{F}) $	422	1642	3662	6482	10102
$ S ^*$	4.34	4.42	4.45	4.46	4.47

Table 7. Computation time on self-dual Fano-plane graphs **Table 8.** Computation time on randomly generated instances

$SDFP$	9	16	23	30	37
BEGK	0.043	1.3	27	590	-
DL	0	0.004	0.22	22	-
BMR	0.001	0.003	0.11	3.4	260
HBC	0	0.023	3.2	540	-
KS	0	0.002	0.032	0.64	26
RS	0	0.001	0.022	0.39	16
DFS	0	0	0.014	0.42	20
$ \mathcal{F} $	15	64	365	2430	16843
$ F ^*$	3.87	6.27	9.63	12.89	15.97
$ dual(\mathcal{F}) $	15	64	365	2430	16843
$ S ^*$	3.87	6.27	9.63	12.89	15.97

p	0.9	0.8	0.7	0.6
BEGK	64	510	-	-
DL	20	210	-	-
BMR	1.8	20	320	-
HBC	0.078	1.9	33	680
KS	3.1	37	290	-
RS	0.12	0.87	6.4	52
DFS	0.093	0.84	6.1	52
cRS	0.13	2.6	29	300
cDFS	0.087	1.8	21	250
$ \mathcal{F} $	1000	1000	1000	1000
$ F ^*$	45.056	39.898	35.024	29.953
$ dual(\mathcal{F}) $	30429	364902	2509943	16809231
$ S ^*$	3.75	4.88	5.94	7.31

Table 9. Computation time on all maximal frequent set from “accidents”

ac	200	150	130	110	90	70	50	30
BEGK	0.54	3.2	8.7	22	87	430	-	-
DL	0.004	0.042	0.28	0.98	4.8	31	270	-
BMR	0.008	0.041	0.074	0.17	2.3	5.7	21	140
HBC	0.004	0.018	0.064	0.16	0.95	3.4	19	170
KS	fail	fail	fail	fail	fail	fail	fail	fail
RS	0.001	0.011	0.02	0.052	0.26	0.78	3.3	32
DFS	0.002	0.013	0.034	0.05	0.23	0.76	3.2	28
cRS	0	0.007	0.027	0.05	0.23	1.4	12	230
cDFS	0.001	0.005	0.019	0.051	0.18	0.95	8.4	170
$ \mathcal{F} $	81	447	990	2000	4322	10968	32207	135439
$ F ^*$	57.48	56.34	72.85	72.23	326.66	326.08	325.31	430.39
$ dual(\mathcal{F}) $	253	1039	1916	3547	7617	17486	47137	185218
$ S ^*$	2.57	3.77	4.25	4.73	5.09	5.70	6.46	7.32

Table 10. Computation time on all maximal frequent set from “BMS-WebView2”

<i>bms2</i>	800	500	400	200	100	50	30	20	10
BEGK	-	-	-	-	-	-	-	-	-
DL	0.87	3.9	5.4	94	-	-	-	-	-
BMR	4.7	18	20	110	380	1000	-	-	-
HBC	0.066	0.2	0.31	1.1	3.5	8.8	23	37	87
KS	fail	fail	fail	fail	fail	fail	fail	fail	fail
RS	0.039	0.12	0.15	0.87	9.2	71	340	800	-
DFS	0.048	0.089	0.15	1.1	13	92	400	950	-
cRS	0.004	0.009	0.015	0.056	0.25	1	4	10	47
cDFS	0.003	0.007	0.012	0.053	0.25	1.1	4.4	12	62
$ \mathcal{F} $	62	152	237	823	2591	6946	17315	30405	74262
$ F ^*$	3338.68	3261.89	3338.18	3337.39	3336.36	3335.91	3335.23	3334.97	3334.19
$ dual(\mathcal{F}) $	4616	16991	15993	89448	438867	1289303	2297560	3064937	4582209
$ S ^*$	1.29	1.88	1.82	1.99	2.01	2.02	2.04	2.07	2.15

on $|\mathcal{F}|$ and $|F|^*$. In particular, RS and DFS are much faster than any other algorithm in almost all instances, up to 10,000 times in some cases. The exceptions are matching graphs and dual matching graphs; both are extreme cases of only few small minimal hitting sets that can be easily found, and of few small hyperedges. Straightforward algorithms are fast for these cases. Also, HBC, cRS and cDFS are faster when the hypergraph is dense. BEGK is the slowest in most instances; algorithms with smaller complexity are not always faster. KS algorithm embodies an idea to unify the isomorphic vertices into one to reduce the number of iterations, but it seems that this is not so much efficient in our experiments. In our extra experiments, such isomorphic vertices exist in only a few iterations, thus the improvement brought about by unifying them would be limited.

Note that in instance M , DL is not slow even though $|dual(\mathcal{F})|$ is very large. This reason would be that for any $S \cap F_i = \emptyset$ for all i and the minimality check would not be required at all. In several instances, BMR is slower than DL, even though it is an improved version. The reason would be that BMR uses up a lot of time in preprocessing.

The following Table 11 lists the average ratios of computation times relative to the case without pruning. The value is the average over all instances in the categories, and smaller values mean more improvement. In some cases the ratio is slightly larger than 1.0, however basically the pruning works well especially for RS. The reason that the pruning is not so efficient for DFS is that DFS already has a pruning method, thus the improvement is limited.

We also evaluated the total memory usage of each algorithm. The memory usage mainly depends on the number of minimal hitting sets, thus we display two extreme cases; dual matching graphs DM and the randomly generated instances p . In the results, all algorithms use a lot of memory when $|\mathcal{F}|$ is large.

Table 11. Reduction ratio of computation time by pruning method

instance	w	l	M	DM	TH	$SDTH$	$SDFP$	p	ac	$bms2$
RS (all)	0.37	0.44	0.73	0.16	1.00	0.20	0.30	0.33	0.34	0.56
DFS (all)	0.98	1.09	1.08	1.03	0.86	1.03	0.46	0.94	0.73	1.01
RS (large)	0.19	0.19	0.96	0.11	0.77	0.12	0.15	0.29	0.17	0.33
DFS (large)	0.96	0.95	1.01	1.00	0.83	1.19	0.68	0.94	0.44	1.00

In particular, DL, BMR and HBC use more memory, since STL library uses a much memory for the sake of making variable operations more efficient. Our algorithm and KS algorithm are quite stable to increasing the number of minimal hitting sets, while the others are quite sensitive. KS uses 2.3 megabytes of memory while ours use 12 megabytes. However, 12 megabytes are used by standard library (libc), thus basically the difference can be ignored.

Table 12. Total memory requirement for dual matching graphs (megabytes) **Table 13.** Total memory requirement for randomly generated instances (megabytes)

DM	20	24	28	32	36	40
BEGK	51	51	58	65	fail	fail
DL	43	45	51	160	580	2300
BMR	21	24	41	110	610	-
HBC	25	76	710	7900	-	-
KS	1.9	3	8	25	94	300
RS	13	13	15	24	66	-
DFS	13	13	15	24	66	-
$ \mathcal{F} $	2^{10}	2^{12}	2^{14}	2^{16}	2^{18}	2^{20}
$ F ^*$	10	12	14	16	18	20
$ dual(\mathcal{F}) $	10	12	14	16	18	20
$ S ^*$	2	2	2	2	2	2

p	0.9	0.8	0.7	0.6
BEGK	51	130	-	-
DL	49	120	-	-
BMR	30	100	660	-
HBC	23	63	850	13000
KS	2.3	2.3	2.3	-
RS	12	12	12	12
DFS	12	12	12	12
cRS	12	12	12	12
cDFS	12	12	12	12
$ \mathcal{F} $	1000	1000	1000	1000
$ F ^*$	45.056	39.898	35.024	29.953
$ dual(\mathcal{F}) $	30429	364902	2509943	16809231
$ S ^*$	3.75	4.88	5.94	7.31

6 Conclusion

We proposed efficient algorithms for solving the dualization problem. The new depth-first search type algorithms are based on reverse search and branch and bound with a restricted search space. We also proposed an efficient minimality condition check method that exploits a new concept called “critical hyperedges”. Computational experiments showed that our algorithms outperform the existing ones in almost all cases, while using less memory even for very large-scale problems with up to millions of hyperedges. In some cases, though, our algorithms

take a long time for the minimality check. Shortening this time will be one of the future tasks. More efficient pruning methods are also an interesting topic of future work.

Acknowledgments

Part of this research is supported by the Funding Program for World-Leading Innovative R&D on Science and Technology, Japan. We thank Khaled Elbassion and Elias C. Stavropoulos for providing us with the programs used in our experiments.

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